Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser, bind **28**, nr. 5

Dan. Mat. Fys. Medd. **28**, no. 5 (1954)

# ON THE EQUIVALENCE OF INVARIANCE UNDER TIME REVERSAL AND UNDER PARTICLE-ANTIPARTICLE CONJUGATION FOR RELATIVISTIC FIELD THEORIES

BY

GERHART LÜDERS

København
i kommission hos Ejnar Munksgaard
1954

Printed in Denmark
Bianco Lunos Bogtrykkeri A/S

For relativistic field theories, in a sense specified in section 2, the invariance under time reversal "of the second kind" (time reversal including particle-antiparticle conjugation) is proved mathematically. Consequently, the postulate of invariance under time reversal ("of the first kind") is, for field theories of this type, completely equivalent to the postulate of invariance under particle-antiparticle conjugation.

---

## Introduction.

It was found by several authors that the postulates of invariance of the laws of nature under time reversal or under particle-antiparticle conjugation[1] allow one to rule out some kinds of couplings which, nevertheless, are in accordance with the postulate of relativistic invariance. Two applications are hitherto known, viz.

(1) Coupling between one Bose field ("mesons") and one Dirac field ("nucleons"). Simultaneous coupling with and without derivatives is forbidden for scalar and pseudovector fields[2,3].

(2) Fermi coupling of four Dirac fields. In a sum of several covariant couplings, the phases of the coupling constants must be the same[4,5].

It is a remarkable fact that both results follow from each of the two postulates. Therefore, one might be led to assume that, quite generally, a relativistic field theory is invariant either

[1] We prefer the term "particle-antiparticle conjugation" (though more lengthy) to the more commonly used denotation "charge conjugation".

[2] As a consequence of particle-antiparticle conjugation, cf. Lüders, G., R. Oehme, and W. E. Thirring, Z. Naturforschung **7** a, 213 (1952); Pais, A., and R. Jost, Phys. Rev. **87**, 871 (1952).

[3] As a consequence of time reversal, cf. Lüders, G., Z. Physik **133**, 325 (1952).

[4] As a consequence of time reversal, cf. Biedenharn, L. C., and M. E. Rose, Phys. Rev. **83**, 459 (1951).

[5] As a consequence of both time reversal and particle-antiparticle conjugation, cf. Tolhoek, H. A. and S. R. de Groot, Phys. Rev. **84**, 151 (1951).

under both transformations or under neither of them. In the present note, the proof of this conjecture will be given. For the sake of simplicity, the considerations are restricted to local field theories constituted by the usual fields of spin 0, 1, and $^1/_2$. The coupling Hamiltonians shall contain no derivatives of the Dirac fields and no higher derivatives than the first of the Bose fields. The theories shall be relativistic in a sense specified in section 2 by the postulates I, II, I′, II′, Ia, IIa. It seems very likely that the result of the considerations, i. e., the equivalence of the two kinds of invariance postulates, holds true also under more general conditions.

In the following, two types of time reversal will appear: the time reversal "of the first kind" which, loosely speaking, consists in a reversal of the motion[1] of all particles, and the time reversal "of the second kind", a simultaneous performance of a proper time reversal and a particle-antiparticle conjugation[2]. (The only type of time reversal which is of relevance for the principle of detailed balance and, perhaps, for the foundation of thermodynamics is that of the first kind[3].) In this paper it is proved that a relativistic field theory (in the sense specified in section 2) is automatically invariant under time reversal of the second kind[4]. For the validity of the proof, one has explicitly to assume that the field theory in question is invariant under reflection in space; a formal reflection in time acts as an intermediate step in the proof.

As the time reversal of the second kind is identical with a simultaneous application of a time reversal of the first kind (i. e., time reversal in the proper sense) and a particle-antiparticle conjugation, a relativistic theory (in the restricted sense of this paper) is either invariant under both operations or under neither of them. Therefore, both of these invariance postulates

[1] In a previous paper (Z. Physik **133**, 325 (1952)), we preferred the term "Bewegungsumkehr" (reversal of motion) to "Zeitumkehr" (time reversal) for this operation.

[2] The different types of time reversal were also considered by J. Tiomno in his Princeton thesis. Further, S. Watanabe (Phys. Rev. 84, 1008 (1951)) uses two types of time reversal; his "standpoint I" corresponds to our time reversal of the second kind, and vice versa.

[3] Coester, F., Phys. Rev. 84, 1259 (1951), Lüders, G., Z. Physik **133**, 325 (1952).

[4] This conjecture was suggested to the writer in a correspondence with B. Zumino, New York.

lead to the same consequences, e. g., the exclusion of some couplings. It seems to be a matter of taste which of these two postulates is considered the more fundamental one.

We shall be concerned with various types of symmetry operations: viz., two types of time reversal, particle-antiparticle conjugation, and reflections in space and in time. These operations will be uniformly treated as substitutions; the prescriptions for these substitutions are summarized in Table 1. In this table, $\varphi(\vec{r})$ means the operator of a spin 0 field, $\dot{\varphi}(\vec{r})$ the operator corresponding to its derivative with respect to time; furthermore $\varphi_k(\vec{r})$ ($k = 1, 2, 3$) are the space components of a spin 1 field, and $\varphi_0(\vec{r})$ is the time component; finally, $\dot{\varphi}_k(\vec{r})$, $\dot{\varphi}_k(\vec{r})$ are the derivatives with respect to the time, and $\psi(\vec{r})$ is the operator of a Dirac field. All operators shall be understood in the Schrödinger representation, where one, of course, has identities of the form

$$\dot{\varphi}(\vec{r}) \equiv i\,[H, \varphi(\vec{r})]. \tag{0.1}$$

Further, one has subsidiary conditions for spin 1 fields with non-vanishing rest mass. In the table, the quantities $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_{1/2}$, $\eta_0$, etc., are multiplicative c-numbers, whereas the symbols $U$, $C$, $T$ mean four-rowed matrices acting on the spinor indices.

In section 1, the mathematical definitions of the two kinds of time reversal and of particle-antiparticle conjugation are summarized. In section 2, the proof is given for the invariance of a relativistic field theory under time reversal of the second kind. In Appendix 2, a lemma on the covariant quantities which can be constructed from Dirac spinors is proved.

## 1. Time reversal of the first and second kind, and particle-antiparticle conjugation.

The *time reversal of the first kind* was formulated for the first time by WIGNER[1]; the substitutions on the field operators, corresponding to this operation, were given in a previous paper[2]. These substitutions are summarized in the second column of Table 1. The most essential prescription is that, according to the

[1] WIGNER, E. P., Göttinger Nachr. math.-phys. Kl. **1932**, 546.
[2] LÜDERS, G., Z. Physik **133**, 325 (1952), in the following quoted as A.

TABLE 1

General comments are given at the end of the introduction and at the beginning of section 1.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Spin | Field operator | Time reversal of the first kind | Particle-antiparticle conj. | Time reversal of the second kind | Reflection in space | Formal reflection in time |
| 0 | $\varphi\,(\vec r)$ | $\varepsilon_0\varphi\,(\vec r)$ | $\eta_0\varphi^*(\vec r)$ | $\delta_0\varphi^*(\vec r)$ | $\xi_0\varphi\,(-\vec r)$ | $\xi_0\varphi\,(\vec r)$ |
| | $\varphi^*(\vec r)$ | $\varepsilon_0^*\varphi^*(\vec r)$ | $\eta_0^*\varphi\,(\vec r)$ | $\delta_0^*\varphi\,(\vec r)$ | $\xi_0^*\varphi^*(-\vec r)$ | $\xi_0^*\varphi^*(\vec r)$ |
| | $\dot\varphi\,(\vec r)$ | $-\varepsilon_0\dot\varphi\,(\vec r)$ | $\eta_0\dot\varphi^*(\vec r)$ | $-\delta_0\dot\varphi^*(\vec r)$ | $\xi_0\dot\varphi\,(-\vec r)$ | $-\xi_0\dot\varphi\,(\vec r)$ |
| | $\dot\varphi^*(\vec r)$ | $-\varepsilon_0^*\dot\varphi^*(\vec r)$ | $\eta_0^*\dot\varphi\,(\vec r)$ | $-\delta_0^*\dot\varphi\,(\vec r)$ | $\xi_0\dot\varphi^*(-\vec r)$ | $-\xi_0^*\dot\varphi^*(\vec r)$ |
| 1 | $\varphi_k(\vec r)$ | $\varepsilon_1\varphi_k(\vec r)$ | $\eta_1\varphi_k^*(\vec r)$ | $\delta_1\varphi_k^*(\vec r)$ | $-\xi_1\varphi_k(-\vec r)$ | $\xi_1\varphi_k(\vec r)$ |
| | $\varphi_k^*(\vec r)$ | $\varepsilon_1^*\varphi_k^*(\vec r)$ | $\eta_1^*\varphi_k(\vec r)$ | $\delta_1^*\varphi_k(\vec r)$ | $-\xi_1^*\varphi_k^*(-\vec r)$ | $\xi_1^*\varphi_k^*(\vec r)$ |
| | $\dot\varphi_k(\vec r)$ | $-\varepsilon_1\dot\varphi_k(\vec r)$ | $\eta_1\dot\varphi_k^*(\vec r)$ | $-\delta_1\dot\varphi_k^*(\vec r)$ | $-\xi_1\dot\varphi_k(-\vec r)$ | $-\xi_1\dot\varphi_k(\vec r)$ |
| | $\dot\varphi_k^*(\vec r)$ | $-\varepsilon_1^*\dot\varphi_k^*(\vec r)$ | $\eta_1^*\dot\varphi_k(\vec r)$ | $-\delta_1^*\dot\varphi_k(\vec r)$ | $-\xi_1^*\dot\varphi_k^*(-\vec r)$ | $-\xi_1^*\dot\varphi_k^*(\vec r)$ |
| | $\varphi_0(\vec r)$ | $-\varepsilon_1\varphi_0(\vec r)$ | $\eta_1\varphi_0^*(\vec r)$ | $-\delta_1\varphi_0^*(\vec r)$ | $\xi_1\varphi_0(-\vec r)$ | $-\xi_1\varphi_0(\vec r)$ |
| | $\varphi_0^*(\vec r)$ | $-\varepsilon_1^*\varphi_0^*(\vec r)$ | $\eta_1^*\varphi_0(\vec r)$ | $-\delta_1^*\varphi_0(\vec r)$ | $\xi_1^*\varphi_0^*(-\vec r)$ | $-\xi_1^*\varphi_0^*(\vec r)$ |
| | $\dot\varphi_0(\vec r)$ | $\varepsilon_1\dot\varphi_0(\vec r)$ | $\eta_1\dot\varphi_0^*(\vec r)$ | $\delta_1\dot\varphi_0^*(\vec r)$ | $\xi_1\dot\varphi_0(-\vec r)$ | $\xi_1\dot\varphi_0(\vec r)$ |
| | $\dot\varphi_0^*(\vec r)$ | $\varepsilon_1^*\dot\varphi_0^*(\vec r)$ | $\eta_1^*\dot\varphi_0(\vec r)$ | $\delta_1^*\dot\varphi_0(\vec r)$ | $\xi_1^*\dot\varphi_0^*(-\vec r)$ | $\xi_1^*\dot\varphi_0^*(\vec r)$ |
| $^1/_2$ | $\psi\,(\vec r)$ | $\varepsilon_{\frac12}U\;\psi\,(\vec r)$ | $\eta_{\frac12}C^*\psi^*(\vec r) = \eta_{\frac12}\bar\psi\,(\vec r)\,\beta C^*$ | $\delta_{\frac12}T^*\psi^*(\vec r) = \delta_{\frac12}\bar\psi\,(\vec r)\,\beta T$ | $\xi_{\frac12}\beta\;\psi\,(-\vec r)$ | $\xi_{\frac12}T\;\psi\,(\vec r)$ |
| | $\psi^*(\vec r)$ | $\varepsilon^*_{\frac12}U^*\psi^*(\vec r)$ | $\eta^*_{\frac12}C\psi\,(\vec r)$ | $\delta^*_{\frac12}T\psi\,(\vec r)$ | $\xi^*_{\frac12}\beta^*\psi^*(-\vec r)$ | $\xi^*_{\frac12}T^*\psi^*(\vec r)$ |
| | $\bar\psi(\vec r)=\psi^*(\vec r)\beta$ | $\varepsilon^*_{\frac12}U^*\bar\psi\;(\vec r)$ | $\eta^*_{\frac12}\beta^*C\psi\,(\vec r)$ | $\delta^*_{\frac12}\beta T\psi\,(\vec r)$ | $\xi_{\frac12}\bar\psi\,(-\vec r)\,\beta$ | $-\xi^*_{\frac12}\bar\psi\,(\vec r)\;T$ |
| Real Bose & Majorana fields | | $\varepsilon_{0,1} = \pm 1$, $\varepsilon_{\frac12} = \pm i$ | $\eta_{0,1,\frac12} = \pm 1$ | $\delta_{0,1} = \pm 1, \delta_{\frac12} = \pm i$ | $\xi_{0,1} = \pm 1$, $\xi_{\frac12} = \pm i$ | $\xi_{0,1} = \pm 1$, $\xi_{\frac12} = \pm i$ |
| c-number | | (c-number)* | c-number | (c-number)* | c-number | c-number |
| Order of factors | | unchanged | unchanged | unchanged | unchanged | reversed |

second line from below, each c-number is to be replaced by its complex conjugated value. Because of the appearance of $i$ in the Schrödinger equation, this leads to the time reversal of the desired character as pointed out in A. From the table, one reads further that, e. g., the operator $\varphi(\vec{r})$ of a scalar or pseudoscalar field is to be replaced by $\varepsilon_0 \varphi(\vec{r})$ and its time derivative $\varphi(\vec{r})$ by $-\varepsilon_0 \varphi(\vec{r})$[1], the quantity $\varepsilon_0$ being a complex number of modulus one[2]. The quantity $\varepsilon_0$ is restricted to $\pm 1$ for real fields. In the same way, $\varepsilon_1$ and $\varepsilon_{1/2}$ have modulus one and $\varepsilon_1 = \pm 1$ for real fields (e. g., $\varepsilon_1 = -1$ for the Maxwell field, c. f., A). The restrictive conditions for Majorana fields, the spin $^1/_2$ analogue to real fields, will be discussed in connection with the particle-antiparticle conjugation. For different kinds of fields one may, of course, choose different factors $\varepsilon$.

. The matrix $U$ entering into the substitutions for Dirac fields was defined in A by the conditions[3,4]

$$U^{-1}\vec{\alpha}^* U = -\vec{\alpha}, \qquad U^{-1}\beta^* U = +\beta, \tag{1.1}$$

and the postulate of unitarity (in order to preserve the commutation relations)

$$UU^{\dagger} = 1. \tag{1.2}$$

This definition is unique up to a factor of modulus one[5].

The *particle-antiparticle conjugation*[6] can be formulated in a quite similar way (third column of the table). The essential feature, which is characteristic of this operation, is the replacement of all field operators by the Hermitian adjoint operators or, in the spin $^1/_2$ case, by a linear combination of these adjoint operators. The quantities $\eta_0$, $\eta_1$, $\eta_{1/2}$ are again complex numbers of modulus one, and $\eta_0$, $\eta_1$ are restricted to $\pm 1$ for real fields

[1] Because of the identity (0.1) and the prescription that $i$ is to be replaced by $-i$, it is only in this way that one can hope to obtain an invariant Hamiltonian $H$.

[2] This is a consequence of the commutation relations, which shall be preserved.

[3] There was a mistake in the corresponding equation (1.9) of A.

[4] $\vec{\alpha}$, $\beta$ are the usual Hermitian Dirac matrices. It is $\gamma_k = i\beta\alpha_k$, $\gamma_4 = \beta$, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\alpha_1\alpha_2\alpha_3$, $\bar{\psi}(\vec{r}) = \psi^*(\vec{r})\beta$; further * = complex conjugated, $^T$ = transposed, $^\dagger$ = Hermitian adjoint of a four-rowed matrix.

[5] Some of the properties of the matrices $U$, $C$, $T$ are summarized in Appendix 1.

[6] Kramers, H. A., Proc. Acad. Sci. Amsterdam **40**, 814 (1937); Pauli, W., Rev. Mod. Phys. **13**, 203 (1941); Schwinger, J., Phys. Rev. **74**, 1439 (1948). The definition of the matrix $C$ in the present paper is identical with that given by Pauli.

(e. g., $\eta_1 = -1$ for the Maxwell field). For real fields one has, of course, to identify $\varphi^*(\vec{r})$, $\varphi_k^*(\vec{r})$, $\varphi_0^*(\vec{r})$ with $\varphi(\vec{r})$, $\varphi_k(\vec{r})$, $\varphi_0(\vec{r})$ in the prescriptions for the substitutions.

The matrix $C$ is defined by

$$C^{-1}\vec{\alpha}^* C = +\vec{\alpha}, \qquad C^{-1}\beta^* C = -\beta \tag{1.3}$$

and

$$C C^\dagger = 1 \tag{1.4}$$

uniquely up to a factor of modulus one. From a comparison of (1.1) and (1.3), one sees that the matrix

$$T = C^* U \tag{1.5}$$

anticommutes with the Dirac matrices $\vec{\alpha}$, $\beta$

$$T^{-1}\vec{\alpha}\, T = -\vec{\alpha}, \qquad T^{-1}\beta\, T = -\beta. \tag{1.6}$$

As $T$ is furthermore a unitary matrix, one may put

$$T = \alpha_x \alpha_y \alpha_z \beta = i\,\gamma_4 \gamma_5, \tag{1.7}$$

which gives a relation between the phase factors of $U$ and $C$.

For a Majorana field[1] the anticommutator between $\psi^*(\vec{r})$ and $\psi(\vec{r}')$ holds unchanged, but one has the subsidiary condition[2]

$$\psi^*(\vec{r}) = C\,\psi(\vec{r}), \qquad \psi(\vec{r}) = C^*\,\psi^*(\vec{r}). \tag{1.8}$$

The operators $\psi(\vec{r})$ and $\psi(\vec{r}')$ do not anticommute any longer but, as a consequence of the foregoing equation, one finds

$$\{\psi_\alpha(\vec{r}), \psi_\beta(\vec{r}')\} = C^*_{\alpha\beta}\,\delta(\vec{r}-\vec{r}'). \tag{1.9}$$

From a special row of the table one sees that, for Majorana fields, $\varepsilon_{1/2}$ is restricted to $\pm i$ and $\eta_{1/2}$ to $\pm 1$.

The *time reversal of the second kind* is obtained if one, first[3], performs a particle-antiparticle conjugation and, subsequently, a time reversal of the first kind. The result of this sequence of

[1] Majorana, E., Nuovo Cimento 14, 171 (1937).
[2] These two equations are compatible because of eq. (A 1.2) in Appendix 1.
[3] This special order of the operations was chosen only to make the prescription unique.

operations is summarized in the fourth column of the table. In all cases, one has

$$\varepsilon\,\eta\,\delta = 1 \qquad (1.10)$$

(Note that $\delta\delta^* = 1$!). The matrix $T$ has been defined by eq. (1.7).

All these symmetry operations can be applied to state vectors by the prescription given in A: Write any state vector $\Psi$ in the form

$$\Psi = \Omega\,\Psi_0, \qquad (1.11)$$

where $\Psi_0$ is the vacuum of the free fields, and perform the substitutions on the creation operator $\Omega$. This prescription is unique in spite of the various possibilities of writing down $\Omega$ as pointed out in A. The prescription just formulated does not lead to contradictions, as all three operations transform creation operators into creation operators and annihilation operators into annihilation operators.

A given field theory is invariant under a symmetry operation if, in the Schrödinger representation, the commutation relations between the field operators, the Hamiltonian H, and possible subsidiary conditions are preserved. One easily checks that this is true for all symmetry operations considered so far if one has non-interacting fields. If we restrict ourselves to interaction Hamiltonians without derivatives, the commutation relations hold unchanged and one has only to examine the interaction part, $H_J$, of the Hamiltonian. That means that we have to investigate whether such a choice of the $c$-number factors in the substitutions can be made so that $H_J$ is simply multiplied by $+1$. Therefore, it is quite clear that a non-trivial problem occurs only if the interaction Hamiltonian is a sum of elementary interactions, where the interaction density, $\mathcal{H}_J(\vec{r})$, is given by a simple product of field operators, supplemented, if necessary, by the Hermitian adjoint expression. If one allows for first derivatives of the Bose fields in the interaction, one has to examine both commutation relations and Hamiltonian; we shall, however, avoid this difficulty by constructing a "nucleus of the interaction representation" which gives commutation relations as in the case without interaction.

A substitution which does not affect $c$-numbers and the order of factors in products can be generated by a canonical transformation. Consequently, the particle-antiparticle transformation is the only transformation considered in this section, which is equivalent to a canonical transformation.

## 2. Invariance of relativistic field theories under time reversal of the second kind.

In order to prove, for relativistic field theories, the invariance under time reversal of the second kind it is, primarily, necessary to give a specified definition of these field theories. Only the case of *non-derivative couplings* will be treated in some detail. The modifications of the considerations for derivative couplings (first order derivatives of Bose fields) are discussed at the end of the section.

A relativistic field theory with non-derivative coupling, constituted by fields of spin 0, 1, $^1/_2$, will be defined by the following two postulates.

I. The commutation relations are identical with those for the free fields.

II. The interaction part, $H_J$, of the Hamiltonian is a Hermitian operator containing no derivatives of the field operators, and the corresponding localized density transforms like a scalar under the orthochronous Lorentz group (including reflections in space, but not in time).

It would certainly be more satisfactory to give a more fundamental definition of relativistic field theories, using the Lagrangian formulation, etc. But, for the present purpose, this would involve the introduction of comparatively complicated general considerations. We are convinced that our results hold also for wider classes of relativistic field theories which are not covered by the postulates I, II and I′, II′, respectively.

In addition to postulate I, we impose a restriction on the relation between different Dirac fields which, in principle, might either commute or anticommute[1]. We explicitly assume

[1] In contrast to opinions occasionally expressed in the literature, there seems not to exist a simple correspondence between theories with anticommuting Dirac fields and those with commuting Dirac fields if one has more than two such fields.

Ia. Kinematically independent Dirac fields *anti*commute.

This is a necessary condition for the general validity of the proof to be given below.

As one of the steps in our proof, viz. the formal reflection in time, leads to a reversal of the order of factors in products, we assume that all products in $\mathcal{H}_J(\vec{r})$ are symmetrized in the same way as they are in the so-called "charge symmetrical" formulation[1].

IIa. Each product of $m$ Bose fields and $2n$ Dirac fields is to be replaced by the sum, divided by $(m+2n)!$, of all permutations of the factors, each of the terms being multiplied by $+1$ or $-1$ for an even or odd permutation of the Dirac fields, respectively.

Making use of postulate Ia, it is seen that this symmetrized product has the simple property that it is multiplied by $(-)^n$ if the order of all factors is reversed.

Before entering into the proof that the invariance under time reversal is a mathematical consequence of the postulates I, Ia, II, IIa, we have first to make clear the way in which Dirac fields can appear in $\mathcal{H}_J$ and then to formulate in more detail the reflection in space. The form in which Dirac fields enter could be restricted by a further postulate, but this is not necessary if we make use of a lemma proved in Appendix 2. According to this lemma, every covariant quantity consisting of products of $2n$ spinors can be represented as a linear combination of products of the well known bilinear covariant quantities

$$\bar{\psi}\psi, \quad i\bar{\psi}\gamma_\mu\psi, \quad i\bar{\psi}\gamma_\mu\gamma_\nu\psi, \quad i\bar{\psi}\gamma_\mu\gamma_5\psi, \quad i\bar{\psi}\gamma_5\psi. \tag{2.1}$$

Consequently, the spin 0 and spin 1 fields are to be combined with these covariant quantities in such a way that one formally obtains a scalar (under the proper Lorentz group) for the interaction density $\mathcal{H}_J(\vec{r})$.

In postulate II, the invariance of $H_J$ under *reflections in space* was stated. The substitutions corresponding to a reflection at the origin of the coordinate system are summarized in column 5

[1] Heisenberg, W., Z. Physik **90**, 209, **92**, 692 (1934); Schwinger, J., Phys. Rev. **74**, 1439 (1948).

of Table 1. The expressions given there are more general than necessary, as one usually restricts $\xi_0$, $\xi_1$ to the values $\pm 1$. Fields with $\xi_{0,1} = +1$ are denoted as proper fields and those with $\xi_{0,1} = -1$ as pseudofields. We shall, however, not restrict $\xi_{1/2}$ apart from having modulus one. One easily checks that the quantities (2.1), constructed from Dirac fields, transform just like ordinary scalars, vectors, tensors of rank two, pseudovectors, and pseudoscalars, if one disregards for the moment the factors $\xi_{1/2}$. Invariance under reflection in space means that the factors $\xi$ can be chosen in such a way that the interaction density, apart from the substitution of $\vec{r}$ by $-\vec{r}$, is simply multiplied by $+1$. Then the integrated interaction Hamiltonian $H_J$ is evidently invariant.

After these preliminaries we are able to give the proof of the invariance under time reversal of the second kind. This proof proceeds in two steps. First, we shall show that a field theory covered by the postulates given above is invariant under a "formal reflection in time", which essentially is the Lorentz transformation of the operators corresponding to a reversal of the direction of the time axis. Subsequently, we shall demonstrate that this reflection in time is equivalent to just the time reversal of the second kind, to which one can go over by the process of Hermitian conjugation.

The substitutions corresponding to the *formal reflection in time* are summarized in column 6 of the table. It is an essential feature of this reflection that the matrix $T$ (eq. (1.7)) plays the role of the corresponding spinor transformation. It should, perhaps, be mentioned that this operation is the only one of all substitutions considered in this paper which cannot be applied to state vectors in a simple manner, as it transforms creation operators into annihilation operators, and vice versa. But this does not matter in our connection, as this reflection enters only as an intermediate step in our proof.

This formal reflection in time is to be accompanied by a reversal of the order of factors in products in order to preserve the commutation relations of the Bose fields. The symmetrization postulate IIa was made in order to have simple behaviour under this reversal of the order of factors. Under this reversal, symmetrized products of operators, among them $2n$ Dirac fields, are multiplied by $(-)^n$ as already pointed out. This result may be expressed by saying that each covariant quantity (2.1) takes up an additional factor $-1$. Consequently, one can easily check that the bilinear covariants transform under formal reflection in time, just as ordinary scalars, vectors, etc., and as it was postulated in the table for Bose fields. Thus, as the interaction density $\mathcal{H}_J(\vec{r})$ is formally a scalar under the proper Lorentz group, and as we choose for the reflection in time the same factors $\xi$ which made the Hamiltonian invariant under reflection in space (cf. Table 1), the field theory in question is also invariant under formal time reflection[1].

In the second step of the proof, we go over from the Hamiltonian reflected in time to that obtained by time reversal of the second kind by means of *Hermitian conjugation*[2]. The Hamiltonian or, more explicitly, its interaction part, was assumed to be a Hermitian quantity (postulate II) and is, therefore, not changed by this operation[3]. On the other hand, all field operators are now replaced by the Hermitian adjoint operators, all $c$-numbers by the complex conjugated numbers, and the order of factors in all products is reversed. Therefore, the original order of factors in all individual, not symmetrized, products is restituted. In this way one gets in fact that Hamiltonian which one can obtain from the original one by a time reversal of the second kind if one chooses the factors $\delta$, entering in the time reversal of the second kind, in such a way that

$$\xi\delta = 1. \tag{2.2}$$

This can be seen from a detailed study of the table. The condition (2.2) can be fulfilled also for real Bose fields and for Majorana fields.

[1] Perhaps it should be emphasized that this reflection in time is treated as a purely formal operation. The problem whether Dirac fields are measurable quantities or not does, therefore, not occur.

[2] This connection between the formal reflection in time and the time reversal of the second kind throws some light on a discrepancy between results obtained by Schwinger (Phys. Rev. 82, 914 (1951)) and by Watanabe (l. c.). The time reversal applied by Schwinger is, in our language, a formal reflection in time; on the other hand, Watanabe pointed out that only time reversal of the second kind (his standpoint I), but not time reversal of the first kind, leads to a determination of the commutation relations.

[3] In this connection, it should perhaps be noted that proofs of the invariance under time reversal or under particle-antiparticle conjugation, for a given field theory, as a rule make use of the Hermitian character of the Hamiltonian.

In this way the proof is finished; the invariance under time reversal of the second kind is a mathematical consequence of the postulates I, II, Ia, IIa. Then, the equivalence of time reversal of the first kind and of particle-antiparticle conjugation is expressed by eq. (1.10), which allows us to go over from one operation to the other.

Finally, the proof has to be extended to couplings involving first *derivatives* of the Bose fields. To this purpose we make a transformation which can be considered as going over to a "nucleus of the interaction representation". We postulate[1] that it is possible to express the operators $\varphi(\vec{r})$, $\dot{\varphi}(\vec{r})$, $\varphi_k(\vec{r})$, etc. by other operators $\tilde{\varphi}(\vec{r})$, $\dot{\tilde{\varphi}}(\vec{r})$, $\tilde{\varphi}_k(\vec{r})$, in such a way that

I'. The commutation relations for the fields $\tilde{\varphi}(\vec{r})$ etc. are formally identical with those for the original free fields (without $\sim$).

II'. The Hermitian Hamiltonian becomes a sum of a part $\tilde{H}_0$, which is formally identical with the free field Hamiltonian for the original fields, and an interaction part $\tilde{H}_J$, the density of which is the 00-component of a relativistic tensor (if expressed by the fields $\tilde{\varphi}(\vec{r})$ etc.!).

We further retain the postulates Ia, IIa, but formulate them now, of course, for the fields $\tilde{\varphi}(\vec{r})$ etc. Then, the whole proof runs as in the case with no derivatives if one applies the substitutions given in the table on the fields $\tilde{\varphi}(\vec{r})$, etc.

Usually, the fields with and without $\sim$ are different only for time derivatives of Bose fields. From

$$\varphi(\vec{r}) = \tilde{\varphi}(\vec{r}) \tag{2.3}$$

and the identity (0.1), it then follows that one actually has the right substitution for $\dot{\tilde{\varphi}}(\vec{r})$ if one simply applies the substitutions on the original fields. But, for the argument of relativistic covariance, the transition to the fields $\tilde{\varphi}(\vec{r})$ etc. is a rather essential step.

[1] Cf. the remarks on the notion of relativistic field theories succeeding postulates I and II.

The author wishes to express his gratitude to Professor NIELS BOHR and to the members of the CERN Theoretical Study Group for discussions on the subject and to Professor W. HEISENBERG for his interest in this work.

*CERN (European Council for Nuclear Research)*
*Theoretical Study Group at*
*the Institute for Theoretical Physics,*
*University of Copenhagen,*
*and*
*Max-Planck-Institut für Physik, Göttingen.*

---

## Appendix 1

### *Properties of the matrices U, C, and T*

The four-rowed matrices $U$ and $C$ are uniquely defined up to a factor of modulus one by eqs. (1.1), (1.2), or (1.3), (1.4), respectively. For the transposed matrices one has[1]

$$U^T = -U, \quad C^T = +C. \tag{A 1.1}$$

From (1.2) or (1.4) and (A 1.1), it follows that

$$UU^* = -1, \quad CC^* = +1. \tag{A 1.2}$$

For the matrix $T$ (eq. (1.7)) one finds

$$UTU^\dagger = CTC^\dagger = -T^*. \tag{A 1.3}$$

It is, according to eq. (1.5),

$$T = C^*U = -U^*C \tag{A 1.4}$$

where use was made of

$$T = T^\dagger \tag{A 1.5}$$

and (A 1.1). Finally, one has

$$TT^\dagger = 1. \tag{A 1.6}$$

[1] Proof either explicitly using a special representation or, more generally, following a method by HAANTJES and PAULI. Cf. PAULI, W., Ann. Inst. H. Poincaré **6**, 137 (1936).

## Appendix 2

*Covariant quantities constructed from Dirac spinors*

*Lemma*: Each covariant quantity constructed from linear combinations of products of $n$ spinors with star ($\psi^*$) and $n$ spinors without star ($\psi$)[1] can be represented as a linear combination of products of $n$ bilinear covariant quantities (eq. (2.1)). It is further possible to build these bilinear covariants for a special pairing, i. e., a special correspondence between the $\overline{\psi} = \psi^* \beta$ and $\psi$ operators so that each pair is connected by a general $\gamma$ matrix (1, $\gamma_\mu$, $\gamma_\mu \gamma_\nu$, etc.).

*Additional remark:* A special case of this lemma was proved by Pauli and Fierz[2]: For four spinors, all scalars constructed from the bilinear covariants for *any* pairing can be expressed as a linear combination of products of bilinear covariants for a *special* pairing. Our lemma is more general in several respects. It asserts that, e. g., any expression $\overset{*}{\psi}_{1\alpha}\, \psi_{2\beta}\, \overset{*}{\psi}_{3\gamma}\, \psi_{4\delta}\, \Gamma_{\alpha\beta\gamma\delta}$ transforming like a scalar can be written as a linear combination of $(\overline{\psi}_1 \psi_2)\,(\overline{\psi}_3 \psi_4)$, $(\overline{\psi}_1 \gamma_\mu \psi_2)\,(\overline{\psi}_3 \gamma_\mu \psi_4)$, etc. Further, the proof is not restricted as regards number of spinors and rank of the tensor to be constructed. The wider validity of our lemma is counterbalanced by the fact that we use a more abstract tool for our proof than Pauli and Fierz did.

*Preliminaries to the proof:* Every finite irreducible representation of the proper Lorentz group can be characterized by two integral or half integral numbers. One denotes such a representation by the symbol $D(j_1, j_2)$. A Dirac spinor (without star as well as with star) transforms according to the representation $D\,(^1/_2, 0) + D\,(0, {}^1/_2)$, which is reducible under the proper Lorentz group, but irreducible under the orthochronous Lorentz group. To the construction of all possible covariant quantities from a pair of Dirac spinors corresponds the decomposition of the Kronecker product of the representations, which can be done according to general rules

$$\left.\begin{aligned} &\left(D(^1/_2, 0) + D(0, {}^1/_2)\right)^2 = \\ &2\, D(0, 0) + 2\, D(^1/_2, {}^1/_2) + D(1, 0) + D(0, 1). \end{aligned}\right\} \quad (A\ 2.1)$$

[1] This assumption means no loss of generality, as the matrix $C$ always allows us to go over from one Dirac spinor to another which behaves like the Hermitian adjoint spinor.

[2] Fierz, M., Z. Physik **104**, 553 (1937).

On the right hand side, one has just the well known bilinear covariants 2 scalars $D(0, 0)$, 2 four vectors $D(^1/_2, ^1/_2)$, and one general six vector $D(1, 0) + D(0, 1)$. Note that we here classify only with respect to the proper Lorentz group, where no difference between proper tensors and pseudo-tensors exists.

*Proof of the lemma*: For 2 $n$ Dirac spinors, the representation

$$\left(D(^1/_2, 0) + D(0, ^1/_2)\right)^{2n} \tag{A 2.2}$$

gives, if decomposed into irreducible constituents,

(1) the linear independent covariant quantities which can be constructed from these $2\,n$ spinors (as (A 2.2) is the $2\,n^{\text{th}}$ power of $D(^1/_2, 0) + D(0, ^1/_2)$),

(2) the covariant quantities which can be constructed from the products of $n$ bilinear covariants for a special pairing (as (A 2.2) is the $n^{\text{th}}$ power of (A 2.1)).

Consequently, the number of linearly independent tensors of given rank which can be constructed from $n$ bilinear covariants for a given pairing is not less than the number of linearly independent tensors of that rank which can be constructed from 2 $n$ spinors.

Indleveret til selskabet den 23. oktober 1953.
Færdig fra trykkeriet den 14. februar 1954.